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THE CONSTRUCTION OF ALL LOCALLY COMPACT EXTENSIONS

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The paper presents one of the ways to construct all the locally compact extensions of a given Tychonoff space T. First, there proved the "local" variant of the Stone–Čech theorem on "completely regular" Riesz spaces $X(T)$ of continuous bounded functions on T with no unit function, in general, but with a collection of local units. In Theorem 1 it is proved that all the functions from $X(T)$ can be "completely regularly" extended on the largest locally compact extension $\beta_X T$. Theorem 3 states, that $\beta_X T$ are presenting, in fact, all the locally compact extensions of T.

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0. Introduction

The paper presents one of the ways to construct all locally compact extensions of a given Tychonoff space T. First, we prove the local variant of the Stone–Čech theorem on "completely regular" Riesz spaces $X(T)$ of continuous bounded functions on T with no unit function, in general, but with a collection of local units. In Theorem 1, we prove that all the functions from $X(T)$ can be "completely regularly" extended on the largest locally compact extension $\beta_X T$. Theorem 3 states, that $\beta_X T$ give, in fact, all locally compact extensions of T. After that, the spaces $X(T)$, producing all the locally compact paracompact extensions of T, are considered (Theorem 4). And finally the criterion for locally compact paracompact "approximation" of T is established (Theorem 5). Theorems 4 and 5 give answers to questions put to the author by Prof. Archangel'skiĭ, to whom the author is very thankful.

In the paper the terms of [1] are used, and some facts from [2] will be essential for the beginning of the paper.

1. The local variant of the Stone–Čech theorem

Let us consider the Riesz space (i.e. vector lattice) $C_b(T)$ of all bounded continuous functions of the Tychonoff space T. A Riesz subspace (i.e. vector subspace and sublattice) $X(T)$ of $C_b(T)$ is called a *completely regular* Riesz space of continuous functions on T if for any closed F and $t \notin F$ there exists $x \in X(T)$, such that $x(F) = 0$ and $x(U_t) = 1$ for some neighborhood U_t of the point t. Further $X(T)$ will always denote a completely regular Riesz space of continuous functions on T.

Let us denote $Dx \equiv \{t \in T \mid x(t) \neq 0\}$ for $x \in X(T)$. The collection $\{a_\alpha, e_\alpha \mid \alpha \in A\}$ of positive functions in $X(T)$ is called *a unit family* on T iff $\bigcup\{Da_\alpha \mid \alpha \in A\} = T$ and $e_\alpha(Da_\alpha) = \{1\}$.

A locally compact space S is said to provide *the completely regular* extension of $X(T)$ iff:

(a) T is a dense subspace of S;

(b) any function $x \in X(T)$ may be extended to a function $x' \in C_b(S)$;

(c) the Riesz space $X(S) \equiv \{x' \mid x \in X(T)\}$ is completely regular on S.

If $\{a_\alpha, e_\alpha\}$ is a unit family in $X(T)$, then S is said to provide the completely regular extension of $X(T)$, preserving the unit family $\{a_\alpha, e_\alpha\}$ iff S satisfies conditions (a)–(c) and condition

(d) $\{a'_\alpha, e'_\alpha\}$ is a unit family on S.

We recall that a vector subspace Y of a Riesz space X is called an ideal in X iff $x \in X$, $y \in Y$ and $|x| \leq |y|$ implies $x \in Y$. Let us consider the $\mathfrak{M}^0_\alpha$ (for any α) of all ideals, maximal among those, not containing the element e_α. According to Theorem 33.4 in [2] each ideal in $\mathfrak{M}^0_\alpha$ is prime. Taking all sets $S^0_\alpha x \equiv \{P \in \mathfrak{M}^0_\alpha \mid x \notin P\}$ as a base of open sets, we make $\mathfrak{M}^0_\alpha$ a topological space. By Theorem 36.4 in [2] the space $\mathfrak{M}^0_\alpha$ is compact. Consider the locally compact subspace $\mathfrak{M}_\alpha \equiv S^0_\alpha a_\alpha$; sets $S_\alpha x \equiv \{P \in \mathfrak{M}_\alpha \mid x \notin P\}$ are the base of open sets in $\mathfrak{M}_\alpha$. The space

$$\mathfrak{M} \equiv \mathfrak{M}(X(T), \{a_\alpha, e_\alpha\}) \equiv \bigcup_\alpha \mathfrak{M}_\alpha$$

is a topological space with the Stone topology if $Sx \equiv \{P \in \mathfrak{M} \mid x \notin P\}$ form the base of open sets in $\mathfrak{M}$. And since $\mathfrak{M}_\alpha \cap Sx = S_\alpha x$, the space $\mathfrak{M}_\alpha$ is a subspace of $\mathfrak{M}$. Besides the sets $Nx \equiv \{P \in \mathfrak{M} \mid x \in P\}$ form the base of closed sets in $\mathfrak{M}$. By the definition of topology for any $P \in \mathfrak{M}$ and any closed set F in $\mathfrak{M}$, such that P is not in F, there exists $x \in X$, for which $P \in Sx$ and $F \subset Nx$.

It is easy to get, that if $P \in \mathfrak{M}_\beta$ and $a_\alpha \notin P$, then $P \in \mathfrak{M}_\alpha$. Theorem 36.3 in [2] implies that $\mathfrak{M}$ is Hausdorff. Since $Sa_\alpha = \mathfrak{M}_\alpha$, the space $\mathfrak{M}_\alpha$ is an open subspace of $\mathfrak{M}$. This means that $\mathfrak{M}$ is locally compact since the $\mathfrak{M}_\alpha$ are locally compact. Using the proof of Theorem 46.1 in [2], we get $\bigcap\{P \mid P \in \mathfrak{M}\} = \{0\}$. Consider the quotient Riesz space X/P with respect to the ideal $P \in \mathfrak{M}$. Let $x \bmod P$ denote the image of x in X/P; let $\langle e_\alpha \bmod P \rangle$ denote the ideal in X/P, consisting of $x \bmod P$, for which there is $m \in \mathbb{N}$ such that $|x| \bmod P \leq m e_\alpha \bmod P$. The Riesz space $\langle e_\alpha \bmod P \rangle$ is Archimedean, that is if $x \bmod P$ and $y \bmod P$ are in $\langle e_\alpha \bmod P \rangle$ and $0 \leq nx \bmod P \leq y \bmod P$ for any

$n \in \mathbb{N}$, then $x \bmod P = 0$. Thus for any $P \in \mathfrak{M}_\alpha$ and any closed F in $\mathfrak{M}$, which does not contain P, there is $x \in X$ such that $x \bmod P = e_\alpha \bmod P$ and $F \subset Nx$.

Any $0 \leqslant x \in X$ and $P \in \mathfrak{M}_\alpha$ are put in correspondence with

$$\tilde{x}(P) = \inf\{\lambda \in \mathbb{R} \mid x \bmod P \leqslant \lambda e_\alpha \bmod P\}.$$

And if $P \in \mathfrak{M}_\beta$, then $e_\alpha \bmod P = e_\beta \bmod P$ and therefore $\tilde{x}(P)$ does not depend on the index α. So, the mapping $\tilde{x}$ from $\mathfrak{M}$ to $\mathbb{R}$ is established: if $x \in X$, then $x_+ \equiv x \vee 0$ and $x_- \equiv (-x) \vee 0$, so $\tilde{x} \equiv \tilde{x}_+ - \tilde{x}_-$.

Using the arguments similar to those in Theorems 44.3 and 44.2 of [2] it is possible to see that the mapping i, which corresponds x to the function $\tilde{x}$, is an injective Riesz homomorphism from X into $C_b(\mathfrak{M})$ i.e. $i(x+y) = ix + iy$, $i\lambda x = \lambda ix$ and $i(x \vee y) = (ix) \vee (iy)$.

Further we will denote $\mathfrak{M}(X(T), \{a_\alpha, e_\alpha\})$ by $\beta(T, X(T), \{a_\alpha, e_\alpha\})$.

Proposition 1. *Let $X(T)$ be a completely regular Riesz space of continuous functions on T and $\{a_\alpha, e_\alpha\}$ be a unit family in $X(T)$. Then:*

(a) *The locally compact space $\beta(T, X(T), \{a_\alpha, e_\alpha\})$ provides the completely regular extension of $X(T)$, preserving the unit family $\{a_\alpha, e_\alpha\}$.*

(b) *$(T, X(T), \{a_\alpha, e_\alpha\})$ is the largest locally compact space, which provides the completely regular extension of $X(T)$, preserving the unit family $\{a_\alpha, e_\alpha\}$, i.e. if S is a locally compact, providing such an extension then there exists the injective continuous mapping σ from S into $\beta(T, X(T), \{a_\alpha, e_\alpha\})$, which is the identity on T.*

Proof. Check, that $T \subset \mathfrak{M}$. For that consider the ideal $M_t \equiv \{x \in X(T) \mid x(t) = 0\}$ in X for $t \in T$. Let us define the injective mapping $\varphi: T \to \mathfrak{M}$, assuming $\varphi(t) \equiv M_t$. If V is an open set in $\mathfrak{M}$, i.e. $V \equiv \bigcup_\nu Sx_\nu$, then $\varphi^{-1}(V) = \bigcup Dx_\nu$ is open in T. Therefore, φ is continuous. Now consider the subspace $T_0 \equiv \{M_t \mid t \in T\}$. Then φ is a homeomorphism from T onto T_0. Further, we consider T and T_0 as identical.

It is easy to check, that $(ix)(\varphi(t)) = x(t)$. So, the function $\tilde{x} \equiv ix$ is the extension of the function $x \in X(T)$ on $\mathfrak{M}$. Let us verify, that T_0 is dense in $\mathfrak{M}$. Let $P \in \mathfrak{M}$ and $P \in Sx \neq \emptyset$. If $Sx \cap T_0 = \emptyset$, then $(ix)(\varphi(t)) = 0$ for any $\varphi(t) \in T_0$. It implies $x(t) = 0$ for any $t \in T$ and $x = 0$, which is not true. Therefore $iX \equiv \tilde{X}$ is a completely regular extension of $X(T)$ on $\mathfrak{M}$. The statement (a) is proved.

Let S provide a completely regular extension of $X(T)$ to $X(S)$, preserving the unit family, i.e. there exists Riesz isomorphism $k: X(T) \to X(S)$, such that $k(x) \mid T = x$. Let us consider the ideal

$$M_s \equiv \{k(x) \in X(S) \mid k(x)(s) = 0\} \quad \text{for } s \in S.$$

The ideal $k^{-1}(M_s)$ is in $\mathfrak{M}(X(T), \{a_\alpha, e_\alpha\})$. Therefore it is possible to define the mapping σ from S into $\beta(T, X(T), \{a_\alpha, e_\alpha\})$, assuming $\sigma(s) \equiv k^{-1}(M_s)$. One can verify that σ is an injective and continuous mapping, which is the identity on T.

Remark. Let locally compact spaces S and R be completely regular extensions of

$X(T)$ and let there exist a continuous injective mapping σ from S into R, which is the identity on T. Then $x' \circ \sigma = x''$, where x' and x'' denote the extensions of $x \in X(T)$ on S and R respectively. Besides, σ is the homeomorphism from S onto subspace $\sigma(S)$ of R.

Let us consider the unit family $\{a_\alpha, e_\alpha\}_{\max}$, containing all the pairs $(a_\alpha, e_\alpha) \subset X(T)$ such that $e_\alpha(Da_\alpha) = \{1\}$. Let us denote $\beta_X T \equiv \beta(T, X(T), \{a_\alpha, e_\alpha\}_{\max})$. Using Proposition 1 we will prove

Theorem 1. *Let $X(T)$ be a completely regular Riesz space of continuous functions on a Tychonoff space T. Then:*

(a) *The locally compact space $\beta_X T$ provides a completely regular extension of $X(T)$.*

(b) *$\beta_X T$ is the largest locally compact space among all the locally compact spaces, providing the completely regular extension of $X(T)$, i.e. if S is a locally compact space, providing such an extension, then there exists an injective continuous mapping σ from S into $\beta_X T$, which is the identity on T.*

Proof. Let S be a completely regular extension of $X(T)$, i.e. there exists a Riesz isomorphism $k: X(T) \to X(S)$, such that $k(x)|T = x$. Let us consider a pair of functions e'_s and a'_s from $X(S)$ (for any $s \in S$) such that $s \in Da'_s$ and $e'_s(Da'_s) = \{1\}$. Then $\{a'_s, e'_s \mid s \in S\}$ is a unit family in $X(S)$ and $\{a_s, e_s \mid s \in S\}$ is a unit family in $X(T)$. Therefore $\{a_s, e_s\} \subset \{a_\alpha, e_\alpha\}_{\max}$. Considering $M_s \equiv \{x' \in X(S) \mid x'(s) = 0\}$ we get

$$k^{-1}(M_s) \in \mathfrak{M}(X(T), \{a_s, e_s\}) \subset \mathfrak{M}(X(T), \{a_\alpha, e_\alpha\}_{\max}).$$

Let us define the mapping σ from S into $\beta_X T$ in the following way: $\sigma(s) \equiv K^{-1}(M_s)$. It is easy to check that σ is an injective continuous mapping which is the identity on T.

Corollary 1. *If $X(T)$ contains a unit function, then $\beta_X T$ is a compactification of T.*

Corollary 2. *If $X(T) = C_b(T)$, then $\beta_X T$ is the usual Stone–Čech compactification βT of the space T.*

2. Extension of $X(T)$ to the functions with compact supports

Let $C_c(S)$ denote the Riesz space of all continuous functions on S with compact supports. Using Theorem 1 we prove

Proposition 2. *Let $X(T)$ be a completely regular Riesz space of continuous functions on T. The following conditions are equivalent:*

(a) *For any $x \in X(T)$ there exists $e \in X(T)$, such that $e(Dx) = \{1\}$.*

(b) *$X(\beta_X T) \subset C_c(\beta_X T)$.*

Proof. Consider $T' \equiv \beta_X T$ and $\beta T'$. Two points $s, t \in \beta T'$ will be considered equivalent (identical), iff $x''(s)=x''(t)$ holds for any $x''\in X(\beta T')$. Let T'' denote the resulting compact space. Consider the subspace $S\subset T''$ such that $s\in S$, if there are a'', e''_1, e''_2 in $X(T'')$, for which $s\in Da''$, $e''_1(Da'')=\{1\}$, $e''_2(De''_1)=\{1\}$. Then $T'\subset S$. The condition (a) implies $\overline{D}_{x''}\subset S$ for any $x''\in X(T'')$. It is easy to verify that $X(T'')|S$ is a completely regular Riesz space on the locally compact space S and an extension of $X(T')$. Since it is impossible to extend over $\beta_X T$, therefore $S=T'$ and $\overline{Dx''}\subset S=T'$. Since $\overline{Dx''}$ is compact in T', then $\overline{Dx'}$ is compact in T'. Therefore the condition (b) is fulfilled. The implication (b) $\Rightarrow$ (a) is obvious.

Call $X(T)$ a complete Riesz space in respect to the uniform convergence of bounded in $X(T)$ sequences, iff $\{x_n\}\subset X(T)$, $y\in X(T)$, $|x_n|\leqslant y$ and $|x_n-x_m|\leqslant(1/n)\mathbf{1}$ for any $m\geqslant n$ implies that there is $x\in X(T)$ for which $|x_n-x|\leqslant(1/n)\mathbf{1}$ for all n. ($\mathbf{1}$ denotes the unit function on T). Using Proposition 2, we prove:

Theorem 2. *Let $X(T)$ be a completely regular Riesz space of continuous functions on a Tychonoff space T. The following conditions are equivalent:*

(a) *$X(T)$ is complete in respect to the uniform convergence of bounded sequences and for any $x\in X(T)$ there is $e\in X(T)$ for which $e(Dx)=\{1\}$.*

(b) $X(\beta_X T)=C_c(\beta_X T)$.

Proof. (a) $\Rightarrow$ (b). Let $T'\equiv\beta_X T$ and $0\leqslant y\in C_c(T')$. Then $X(T')\subset C_c(T')$ and $K\equiv\overline{Dy}$ is compact in T'. Therefore, there exists $e'\in X(T')$ such that $e'(t)\leqslant 1$ for any $t\in T'$ and $e'(K)=\{1\}$. Let us consider the compact

$$K_n\equiv\left\{t\in T'\,\middle|\,y(t)\geqslant\frac{1}{n}\right\}\subset K.$$

There is a finite set of functions a'_i and x'_i such that $K_n\subset\bigcup Da'_i$, $x'_i\leqslant y$ and $y(s)-x'_i(s)\leqslant 2/n$ for any $s\in Da'_i$. Let us consider $x'_n\equiv\sup x'_i\leqslant y$. Then $y(s)-x'_n(s)\leqslant 2/n$ holds for any $s\in T'$. The sequence $\{x'_n\}$ is fundamental and bounded. Therefore $y\in X(T')$.

The implication (a) $\Rightarrow$ (b) is checked in a standard way.

3. The construction of all locally compact extensions.

We shall say that the completely regular Riesz space $X(T)$ *has subcompact supports*, if the condition (a) of Theorem 2 holds in $X(T)$.

Using Theorems 1 and 2 we get the main

Theorem 3. *Let T be a Tychonoff space. Then:*

(1) *For any completely regular Riesz space $X(T)$ of continuous functions with subcompact supports there exists the only locally compact extension $\beta_X T$ for which:*

(a) *$\beta_X T$ provides a completely regular extension of $X(T)$;*

(b) *$\beta_X T$ is the largest locally compact extension among all the locally compact spaces, providing a completely regular extension of $X(T)$;*

(c) $X(\beta_X T) = C_c(\beta_X T)$.

(2) *For any locally compact extension γT there exists the completely regular Riesz space $X(T)$ of continuous functions with subcompact supports, for which $\beta_X T = \gamma T$;*

(3) *if $X(T)$ and $Y(T)$ are completely regular Riesz spaces of continuous functions with subcompact supports, for which $\beta_X T$ and $\beta_Y T$ are homeomorphic under the homeomorphism identical on T, then $X(T)$ and $Y(T)$ coincide.*

Proof. Let γT be come locally compact extension and $X(T) \equiv C_c(\gamma T)|T$. Then $X(T)$ is a completely regular Riesz space of continuous functions with subcompact supports. Applying Theorem 1 we get $T \subset \gamma T \subset \beta_X T$. One can check that $X(\gamma T) = C_c(\gamma T)$ implies $\gamma T = \beta_X T$.

Let $h: \beta_Y T \to \beta_X T$ be a homeomorphism which is the identity on T. We consider the isomorphism $k: X(\beta_X T) \to Y(\beta_Y T)$, defined as $kx' \equiv x' \circ h$. Let $i: X(T) \to X(\beta_X T)$ and $j: Y(\beta_Y T) \to Y(T)$ be the canonical Riesz isomorphism of extension and retraction. Then $\alpha \equiv j \circ k \circ i$ is an identical map from $X(T)$ onto $Y(T)$. So Theorem 3 is proved.

Corollary 1. *If in the statement of Theorem 3 $X(T)$ with a unit function, then all the compact extensions (i.e., compactifications) of T are obtained.*

The construction of all compactifications of a Tychonoff space was originally delivered by Y. M. Smirnov [3] and then (in a different way) by P.S. Alexandroff and V.I. Ponomarev [4].

Also from the Theorem the following characteristic of locally compact space is obtained.

Corollary 2. *The following statements are equivalent:*

(a) *T is locally compact.*

(b) *There exists a completely regular Riesz space $X(T)$ of continuous functions on T with subcompact supports which does not have a completely regular extension to any locally compact extension of T.*

4. The construction of all locally compact paracompact extensions

Call $\{e_\mu^n \in X(T) \mid \mu \in M, n \in \mathbb{N}\}$ *a star countable unit family* on T, iff

$$T = \bigcup_\mu \Big(\bigcup_n De_\mu^n\Big), \Big(\bigcup_n De_\mu^n\Big) \cap \Big(\bigcup_n De_\nu^n\Big) = \emptyset$$

and $e_\mu^{n+1}(De_\mu^n) = \{1\}$. The star-countable unit family $\{e_\mu^n\}$ is called majorizing iff for any $x \in X(T)$ there is a finite set of indexes μ_i, for which $(\sup e_{\mu_i}^n)(Dx) = \{1\}$ for some n.

Proposition 3. *Let T be a Tychonoff space. Then for any completely regular Riesz space $X(T)$ of continuous functions with subcompact supports and with a star-countable unit family $\{e_\mu^n \in X(T)\}$, the space $\beta(T, X(T), \{e_\mu^n, e_\mu^{n+1}\})$ is a paracompact. Each locally compact paracompact extension of T is constructed in such a way.*

Proof. Let us denote $T' \equiv \beta_X T$. Then $X(T') \equiv C_c(T')$. It is clear that

$$\bigcup_n D(e_\mu^n)' \Big) \cap \Big(\bigcup_n D(e_\nu^n)' \Big) = \emptyset,$$

where $(e_\mu^n)'$ is the extension of e_μ^n to T'. The subspace $S_\mu \equiv \bigcup_n D(e_\mu^n)' = \bigcup_n \overline{D(e_\mu^n)'}$ is a locally compact space, which is a union of a countable set of compacts. Therefore the subspace $S \equiv \bigcup_\mu S_\mu$ is a paracompact space. One can check that $S = \beta(T, X(T), \{e_\mu^n, e_\mu^{n+1}\})$.

Let πT is some locally compact paracompact extension. Applying [5] we get $\pi T = \bigcup_\mu S_\mu$, where S_μ are open disjoint subsets, $S_\mu = \bigcup_\mu U_\mu^n$, U_μ^n are open subsets, $\overline{U_\mu^n}$ are compact subsets and $\overline{U_\mu^n} \subset U_\mu^{n+1}$. So we get the family $\{f_\mu^n \in C_c(\pi T) \mid \mu \in M, n \in \mathbb{N}\}$, for which $S_\mu = \bigcup_n Df_\mu^n$ and $f_\mu^{n+1}(Df_\mu^n) = \{1\}$. Let us consider a completely regular Riesz space $X(T) \equiv C_c(\pi T)|T$ of continuous functions with subcompact supports. Let $e_\mu^n \equiv f_\mu^n | T$ One can check that $\pi T = \beta(T, X(T), \{e_\mu^n, e_\mu^{n+1}\})$. So the proposition is proved.

Theorem 4. *Let T be a Tychonoff space. Then:*

(1) *For any completely regular Riesz space $X(T)$ of continuous functions with subcompact supports and with a star-countable majorizing unit family the space $\beta_X T$ is paracompact.*

(2) *For any locally compact paracompact extension πT there exists the only completely regular Riesz space $X(T)$ of continuous functions with subcompact support and with star-countable majorizing unit family, for which $\beta_X T = \pi T$.*

Proof. Proposition 3 implies $T' \equiv \beta(T, X(T), \{e_\mu^n, e_\mu^{n+1}\})$ is paracompact. One can check that the space $T'' \equiv \beta_X T$ provides the completely regular extension of functions from $X(T)$, preserving the unit family $\{e_\mu^n, e_\mu^{n+1}\}$. So Proposition 1 implies $T \subset T'' \subset T'$. But according to Theorem 1 $T \subset T' \subset T''$. Therefore $T'' = T'$ and T'' is paracompact. Using Proposition 3 the second statement of the theorem is obtained quite easily.

From this theorem the following characteristic of locally compact paracompact space is obtained.

Corollary. *The following statements are equivalent:*

(a) *T is locally compact paracompact.*

(b) *There exists a completely regular Riesz space $X(T)$ of continuous functions on T with subcompact supports and with star-countable unit family, which does not have a completely regular continuation to any locally compact paracompact extension of T.*

5. The characterization of the spaces, providing locally compact paracompact approximation

Theorems 3 and 4 imply:

Theorem 5. *Let T be a Tychonoff space. The following statements are equivalent:*

(1) *For any locally compact extension γT of the space T there exists the locally compact paracompact extension πT for which $T \subset \pi T \subset \gamma T$.*

(2) *In any completely regular Riesz space $X(T)$ of continuous functions with subcompact supports there is a star-countable unit family.*

Proof. (2)$\Rightarrow$(1). Let us consider any locally compact extension γT and $X(T) \equiv C_c(\gamma T)|T$. Then there is a star-countable unit family $\{e_\mu^n\} \subset X(T)$. Proposition 3 implies that the space $\pi T \equiv \beta(T, X(T), \{e_\mu^n, e_\mu^{n+1}\})$ is a paracompact. Using the proof of Theorem 3 we get $\beta_X T = \gamma T$. So according to Theorem 1 $T \subset \pi T \subset \gamma T$.

(1)$\Rightarrow$(2). Let $X(T)$ be any completely regular Riesz space of continuous functions with subcompact supports. Let us consider $\gamma T \equiv \beta_X T$. By (1) there exists locally compact paracompact space πT, for which $T \subset \pi T \subset \gamma T$. Since πT is paracompact, [5] implies $\pi T = \bigcup_\mu S_\mu$, where S_μ are open and disjoint in πT, $S_\mu = \bigcup_n U_\mu^n$, U_μ^n are open in πT, $\overline{U_\mu^n}$ are compact and $\overline{U_\mu^n} \subset U_\mu^{n+1}$. Since $X(\gamma T) = C_c(\gamma T)$ is completely regular, there are $f_\mu^n \in X(\gamma T)$ such that $S_\mu = \bigcup_n Df_\mu^n$ and $f_\mu^{n+1}(Df_\mu^n) = \{1\}$. Let us denote $e_\mu^n \equiv f_\mu^n | T$. Then $\{e_\mu^n \mid \mu \in M, n \in \mathbb{N}\}$ is a star-countable family. The theorem is proved.

Corollary. *If in a normal space the condition 2 holds, then it is strongly paracompact.*

Proof. Let $\{U_\alpha\}$ be an open cover of T. Let us consider another cover $\{V_\beta\}$ which members are finite unions of U_α. Let us consider the Riesz space $X(T)$ of all functions $f \in C_b(T)$ such that $\overline{Df} \subset V_\beta$ for some β. By (2) there exists a star-countable unit family $\{e_\mu^n\}$ in $X(T)$. Since $De_\mu^n \subset V_\beta = \bigcup_{i=1}^k U_{\alpha_i}$, let us consider the cover, consisting of sets of following kind: $De_\mu^n \cap U_{\alpha_i}$. This cover is star-countable and a refinement of $\{U_\alpha\}$. Therefore T is strongly paracompact.

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